

Decomposing the vertex set of a hypercube into isomorphic subgraphs

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Abstract

Let G be an induced subgraph of the hypercube Q_k for some k . We show that if $|G|$ is a power of 2 then, for sufficiently large n , the vertex set of Q_n can be partitioned into induced copies of G . This answers a question of Offner. In fact, we prove a stronger statement: if X is a subset of $\{0, 1\}^k$ for some k and if $|X|$ is a power of 2, then, for sufficiently large n , $\{0, 1\}^n$ can be partitioned into isometric copies of X .

1 Introduction

A famous theorem of Wilson [12] states that, for any finite graph H and for any sufficiently large integer n which satisfies certain divisibility conditions, the edges of the complete graph K_n can be covered by disjoint copies of H . Such a cover is called an H -decomposition of K_n . The divisibility conditions required by Wilson's theorem are obviously necessary for an H -decomposition of K_n to exist: $\binom{n}{2}$ must be divisible by $e(H)$ and $n - 1$ must be divisible by the highest common factor of the degrees of the vertices of H . Therefore, as long as we are only interested in large n , Wilson's theorem tells us exactly when K_n admits an H -decomposition. On the other hand, the general question of determining whether an arbitrary graph G has a H -decomposition is very difficult, and various special cases of this question have attracted significant attention.

In this paper we examine a related question: we are concerned with partitioning the vertices – not edges – of a given graph G into copies of H . More precisely, for finite graphs G, H , we say that a set $A \subset V(G)$ is an H -set if the induced subgraph $G[A]$ is isomorphic to H . We consider the following question: can $V(G)$ be partitioned into H -sets?

In contrast to Wilson's theorem, this question is not interesting in the case where G is a complete graph: obviously, $V(K_n)$ can be partitioned into H -sets if and only if $H = K_m$ where m divides n . Instead, we focus on the case where G is the hypercube Q_n , that is,

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the graph with vertex set $\{0, 1\}^n$ where two n -tuples are adjacent if and only if they differ in precisely one entry.

Let H be a finite graph and let n be large. Can we quickly determine whether $V(Q_n)$ can be partitioned into H -sets? Of course, there is an obvious necessary divisibility condition: $|H|$ must be a power of 2. Moreover, this condition alone is not sufficient because H may not be isomorphic to any induced subgraph of any hypercube Q_n . For example, H could be a non-bipartite graph or, say, it could be a bipartite graph, of size a power of 2, that contains $K_{3,2}$ as a subgraph. Note $K_{3,2}$ is not a subgraph of any Q_n since any two vertices that are distance 2 apart in Q_n are joined by precisely two paths of length 2. Therefore, we should require H to be an induced subgraph of some hypercube. Offner [9] considered this problem in connection with coding theory. He asked if this condition together with the divisibility condition is sufficient.

Question 1 (Offner). *Let H be an induced subgraph of Q_k for some k and suppose that $|H|$ is a power of 2. Must it be true that, for any sufficiently large n , $V(Q_n)$ can be partitioned into H -sets?*

This question bears resemblance to the celebrated work of Hamming [5] on error-correcting codes. Indeed, a perfect single-error-correcting code is a partition of $V(Q_n)$ into $K_{1,n}$ -sets. Hamming showed that such a partition exists if and only if a natural divisibility condition is satisfied, namely, if $n = 2^r - 1$ for some r . Much later, Rogers [10, 11] asked if it is possible to partition the vertices of Q_n into antipodal paths, subject to the same divisibility condition. Here an *antipodal path* is a path of length n which starts and ends at two diagonally opposite vertices of Q_n . Rogers' question was answered by Ramras [10], who proved the following more general result: if $n = 2^r - 1$ and if T is a tree on $n + 1$ vertices which is an induced subgraph of Q_n , then $V(Q_n)$ can be partitioned into isometric copies of T .

On the other hand, there is an important difference between Offner's question and the work of Hamming and Ramras: in Offner's question H is fixed and n can be taken to be large. In fact, Offner's question is more closely related to the following two conjectures coming from different areas of combinatorics. One was proposed by Chalcraft [7, 8].

Conjecture 2 (Chalcraft). *Let T be a non-empty finite subset of \mathbb{Z}^k for some k , where \mathbb{Z}^k is treated as a subspace of the metric space \mathbb{R}^k . Then, for sufficiently large n , the space \mathbb{Z}^n can be partitioned into isometric copies of T .*

Another related conjecture was proposed by Lonc [6].

Conjecture 3 (Lonc). *Let P be a poset with a greatest and a least element. If $|P|$ is a power of 2, then, for sufficiently large n , the Boolean lattice $2^{[n]}$ can be partitioned into sets, each of which induces a poset isomorphic to P .*

These conjectures were recently solved: Gruslys, Leader and Tan [3] confirmed Chalcraft's conjecture and Gruslys, Leader and Tomon [4] confirmed Lonc's conjecture. In this

paper we combine new ideas with tools developed by these authors to give a positive answer to Offner's question.

Theorem 4. *Let H be an induced subgraph of Q_k for some k . If $|H|$ is a power of 2, then there exists a positive integer n such that the vertices of Q_n can be partitioned into H -sets.*

Of course, if the result holds for n , then it holds for all $n' \geq n$. Therefore, Theorem 4 answers Question 1.

2 Overview of the proof

It turns out that, in order to prove Theorem 4, it is convenient to view the hypercube Q_n as the metric space $\{0,1\}^n$ where the distance between any two points $x, y \in \{0,1\}^n$, denoted $d(x, y)$, is equal to the number of entries where x and y are different. With this definition, $d(x, y)$ equals 1 if and only if x and y are adjacent vertices of Q_n . If H is an induced subgraph of Q_k , then we can identify H with a subset of $\{0,1\}^k$. For any $n \geq k$, we say that a set $X \subset \{0,1\}^n$ is an *isometric copy of H* if there exists an isometry $\phi : \{0,1\}^k \rightarrow \{0,1\}^n$ which maps H to X . Clearly, any isometric copy of H in $\{0,1\}^n$ is an H -set, but an H -set need not be an isometric copy of H .

We deduce Theorem 4 from the following slightly stronger result.

Theorem 5. *Let X be a subset of $\{0,1\}^k$ for some k . If $|X|$ is a power of 2, then there exists a positive integer n such that $\{0,1\}^n$ can be partitioned into isometric copies of X .*

A major tool in our proof of Theorem 5 is a theorem of Gruslys, Leader and Tomon [4]. Roughly speaking, their theorem says that, if we are trying to partition an arbitrarily large power A^n of some set A into copies of some given set (which is exactly what we are doing here), then it is enough to construct two specific covers of a large power of A , called an ' r -partition' and a ' $(1 \bmod r)$ -partition'. We will now define these covers and we will see that it is easier to construct them than to directly construct a partition of $\{0,1\}^n$. We will state the aforementioned theorem of Gruslys, Leader and Tomon (Theorem 8) after we have given the necessary definitions.

Let \mathcal{F} be a family of subsets of a set S . A *weight function on \mathcal{F}* is an assignment of non-negative integer weights to the members of \mathcal{F} . For a weight function $w : \mathcal{F} \rightarrow \mathbb{Z}^{\geq 0}$ and an element $x \in S$, the *multiplicity of x for w* is defined to be the sum of weights assigned to the members of \mathcal{F} that contain x .

Let \mathcal{F} and S be as above and let r be a positive integer. We say that \mathcal{F} *contains an r -partition of S* if there exists a weight function on \mathcal{F} for which every element of S has multiplicity r . Moreover, we say that \mathcal{F} *contains a $(1 \bmod r)$ -partition of S* if there exists a weight function on \mathcal{F} for which the multiplicity of every element of S is congruent to 1 (mod r); it is not required that all elements of S have the same multiplicity as long as they all satisfy the required congruence.

Trivially, \mathcal{F} contains a 1-partition of S if and only if S can be partitioned into members of \mathcal{F} . Furthermore, if \mathcal{F} contains a 1-partition of S , then \mathcal{F} also contains an r -partition and a $(1 \bmod r)$ -partition of S , for any positive integer r . Therefore, the property of containing an r -partition and a $(1 \bmod r)$ -partition for some r is weaker than that of containing a genuine partition. However, we will be able to apply the aforementioned theorem of Gruslys, Leader and Tomon to obtain the stronger property from the weaker one. The statement of this theorem requires one more technical definition, and we postpone it until the end of the section. Instead, we will now discuss how to construct an r -partition and a $(1 \bmod r)$ -partition of $\{0, 1\}^n$ into isometric copies of some $X \subset \{0, 1\}^k$ whose size is a power of 2. First, we have to decide what value of r to use. It turns out that the right choice is $r = |X|$.

Observation 6. *Let X be a non-empty subset of $\{0, 1\}^k$ for some positive integer k . Then, for any $n \geq k$, the family of isometric copies of X in $\{0, 1\}^n$ contains a $|X|$ -partition of $\{0, 1\}^n$.*

Proof. Let $n \geq k$ be given. We fix one isometric copy of X in $\{0, 1\}^n$, which we denote by Y . Under addition modulo 2, for any $p \in \{0, 1\}^n$, the set $Y + p = \{y + p : y \in Y\}$ is a subset of $\{0, 1\}^n$. Moreover, it is an isometric copy of X .

By symmetry, all elements of $\{0, 1\}^n$ are contained in $Y + p$ for the same number of choices of p . By double counting, this number must equal $2^n |Y| / 2^n = |X|$. Therefore, the sets $Y + p$, where $p \in \{0, 1\}^n$, form a $|X|$ -partition of $\{0, 1\}^n$. \square

Constructing a $(1 \bmod |X|)$ -partition is rather more difficult, but also possible.

Lemma 7. *Let X be a non-empty subset of $\{0, 1\}^k$ for some positive integer k , and let r be a power of 2. Then there exists an integer $n \geq k$ such that the family of all isometric copies of X in $\{0, 1\}^n$ contains a $(1 \bmod r)$ -partition of $\{0, 1\}^n$.*

Although we are only going to use this lemma with $r = |X|$, we state it with r being any power of 2. This small detail will allow us to prove this lemma by induction, which we do in Section 3.

We now turn to giving the final definitions needed for the statement of Theorem 8. Let S be a set. Here and in the rest of the paper, for any non-negative integers m, n , we identify $S^m \times S^n$ with S^{m+n} . Therefore, for any $x \in S^m, y \in S^n$, we treat (x, y) as an element of S^{m+n} . Furthermore, for any set $X \subset S^n$ and any permutation $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$, we define $\pi(X)$ to be the image of X after permuting the coordinates according to π . In other words, $\pi(X) = \{(x_{\pi(1)}, \dots, x_{\pi(n)}) : (x_1, \dots, x_n) \in X\}$. Finally, for any sets $Y \subset S^m, Z \subset S^n$ with $m \leq n$, we say that Z is a *copy* of Y if $Z = \pi(Y \times \{z\})$ for some $z \in S^{n-m}$ and some permutation $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$.

Theorem 8 (Gruslys, Leader and Tomon [4]). *Let \mathcal{F} be a family of subsets of a finite set S . If, for some positive integer r , \mathcal{F} contains an r -partition and a $(1 \bmod r)$ -partition*

of S , then there exists a positive integer n such that S^n can be partitioned into copies of members of \mathcal{F} .

Theorem 8 is the only result that we use without proof. We will now explain how 6, Lemma 7, and Theorem 8 imply Theorem 5.

Proof of Theorem 5. Let X be a subset of $\{0, 1\}^k$ such that $|X|$ is a power of 2. It follows from Lemma 7 that there exists a positive integer $m \geq k$ such that the family of isometric copies of X in $\{0, 1\}^m$ contains a $(1 \bmod |X|)$ -partition of $\{0, 1\}^m$. By 6, the family of isometric copies of X in $\{0, 1\}^m$ also contains a $|X|$ -partition of $\{0, 1\}^m$. Therefore, it follows from Theorem 8 with $S = \{0, 1\}^m$ that there exists a positive integer n such that $\{0, 1\}^{mn}$ can be partitioned into copies of sets which are isometric copies of X in $\{0, 1\}^m$. However, a copy of an isometric copy of X is itself an isometric copy of X , so we are done. \square

3 Constructing a $(1 \bmod r)$ -partition of $\{0, 1\}^n$

Here we prove Lemma 7. This section is the heart of the paper: it is the key new ingredient beyond the ideas of [3] and [4]. First, we introduce some convenient notation. For any set $A \subset \{0, 1\}^n$, we define

$$\begin{aligned} A_+ &= \{a \in \{0, 1\}^{n-1} : (a, 1) \in A\}, \\ A_- &= \{a \in \{0, 1\}^{n-1} : (a, 0) \in A\}. \end{aligned}$$

Proof of Lemma 7. Fix $r = 2^d$. We will use induction on k . If $k = 1$, then X is either a single point or the whole $\{0, 1\}$, and so the conclusion holds with $n = 1$.

We now suppose that $k \geq 2$. At least one of the sets X_+ and X_- is not empty, so we may assume without loss of generality that $X_- \neq \emptyset$. Since X_- is a subset of $\{0, 1\}^{k-1}$, the induction hypothesis implies the existence of a positive integer m such that the family of isometric copies of X_- in $\{0, 1\}^m$ contains a $(1 \bmod r)$ -partition of $\{0, 1\}^m$. Moreover, we note that, for every set $A \subset \{0, 1\}^m$ which is an isometric copy of X_- , there exists a set $B \subset \{0, 1\}^{m+1}$ which is an isometric copy of X and which satisfies $B_- = A$. Therefore, it is possible to define a weight function on the family of isometric copies of X in $\{0, 1\}^{m+1}$ in such a way that the multiplicity of every element of $\{0, 1\}^m \times \{0\}$ is congruent to 1 (mod r). We do not impose any conditions on the multiplicities of elements of $\{0, 1\}^m \times \{1\}$. For convenience, we denote that the multiplicity of any $x \in \{0, 1\}^m \times \{1\}$ is congruent to $f(x)$ (mod r).

We will prove that the conclusion of Lemma 7 holds with $n = m + d + 1$. Let $x, y \in \{0, 1\}^{d+1}$ be two elements that differ in exactly two entries. There exists an element $z \in \{0, 1\}^{d+1}$ that differs from both x and y in exactly one entry. Then $\{0, 1\}^m \times \{x, z\}$ is an isometric copy of $\{0, 1\}^{m+1}$, while $\{0, 1\}^m \times \{x\}$ and $\{0, 1\}^m \times \{z\}$ are isometric

copies of $\{0,1\}^m$. Therefore, there exists an isometry $\phi : \{0,1\}^m \times \{x,z\} \rightarrow \{0,1\}^{m+1}$ which maps $\{0,1\}^m \times \{x\}$ to $\{0,1\}^m \times \{0\}$ and $\{0,1\}^m \times \{z\}$ to $\{0,1\}^m \times \{1\}$. Hence, it is possible to assign integer weights to the isometric copies of X in $\{0,1\}^m \times \{x,z\}$ so that the multiplicity of every element of $\{0,1\}^m \times \{x\}$ is congruent to 1 (mod r), and the multiplicity of any $p \in \{0,1\}^m \times \{z\}$ is congruent to $f(\phi(p))$ (mod r). We denote the resulting weight function by w' .

The restriction of ϕ to $\{0,1\}^m \times \{z\}$ maps this set isometrically onto $\{0,1\}^m \times \{1\}$. This map extends to an isometry $\{0,1\}^m \times \{y,z\} \rightarrow \{0,1\}^{m+1}$. Therefore, we can assign integer weights to the isometric copies of X in $\{0,1\}^m \times \{y,z\}$ in such a way that every element of $\{0,1\}^m \times \{y\}$ has multiplicity congruent to 1 (mod r), and any $p \in \{0,1\}^k \times \{z\}$ has multiplicity congruent to $f(\phi(p))$ (mod r). We denote the resulting weight function by w'' .

Although, technically, the weight functions w', w'' are only defined on isometric copies of X in, respectively, $\{0,1\}^m \times \{x,z\}$ and $\{0,1\}^m \times \{y,z\}$, we may suppose that they are defined and equal to 0 on the other isometric copies of X in $\{0,1\}^n$. Then $w' + (r-1)w''$, which we denote by $w_{x,y}$, is a weight function on the family of all isometric copies of X in $\{0,1\}^n$. Moreover, for any $p \in \{0,1\}^n$, the multiplicity of p for $w_{x,y}$ is congruent to

$$\begin{cases} 1 \pmod{r} & \text{if } p \in \{0,1\}^m \times \{x\}, \\ -1 \pmod{r} & \text{if } p \in \{0,1\}^m \times \{y\}, \\ 0 \pmod{r} & \text{otherwise.} \end{cases}$$

The existence of the weight functions $w_{x,y}$ simplifies our problem in the following way. Let us view $\{0,1\}^n$ as the product set $\{0,1\}^m \times \{0,1\}^{d+1}$. Given two elements $x, y \in \{0,1\}^{d+1}$ with $d(x,y) = 2$, we identify the pair (x,y) with both the directed edge \overrightarrow{xy} on $\{0,1\}^{d+1}$ and the weight function $w_{x,y}$. Now, our aim is to find a family (allowing repetitions) of directed edges on $\{0,1\}^{d+1}$, whose every member joins two elements of $\{0,1\}^{d+1}$ that are distance 2 apart, and such that for any $v \in \{0,1\}^{d+1}$ the difference between the in-degree and out-degree of v is congruent to 1 (mod p). Indeed, such a family of directed edges corresponds to a weight function for which every element of $\{0,1\}^n$ has multiplicity congruent to 1 (mod r).

We will now construct a family of directed edges with the desired properties. Fix vertices $x^* = (0, \dots, 0) \in \{0,1\}^{d+1}$ and $y^* = (1, 0, \dots, 0) \in \{0,1\}^{d+1}$. Note that, for any vertex $v \in \{0,1\}^{d+1}$, there exists a directed path starting from x^* or y^* and ending at v with the property that any two consecutive vertices on this path differ in exactly two entries. Such a path increases the difference between the in-degree and the out-degree of v by 1, decreases this parameter of its starting point (x^* or y^*) by 1 and does not change the value of this parameter for any other vertex. Now, for any vertex $v \in \{0,1\}^{d+1} \setminus \{x^*\}$ with an even number of 1's, select one such path from x^* to v . Similarly, for any $v \in \{0,1\}^{d+1} \setminus \{y^*\}$ with an odd number of 1's, select one such path from y^* to v . Let us combine all of these paths together to obtain a family of directed edges. It is clear that for any $v \in \{0,1\}^{d+1} \setminus \{x^*, y^*\}$

the difference between the in-degree and the out-degree of v is equal to 1. Moreover, excluding x^* , there are $2^d - 1$ vertices in $\{0, 1\}^{d+1}$ with an even number of 1's. Therefore, the difference between the in-degree and the out-degree of x^* is $-(2^d - 1) \equiv 1 \pmod{r}$. Similarly, the difference between the in-degree and the out-degree of y^* is also congruent to 1 \pmod{r} . This finishes the proof. \square

4 Concluding remarks and open problems

The statement of Theorem 5 is very similar to that of Chalcraft's conjecture. Indeed, the only difference is that, instead of an infinite space \mathbb{Z}^n , here we are dealing with a finite hypercube $\{0, 1\}^n$. However, the results are, in fact, significantly different.

To illustrate this claim, we note that not every sensible finite version of Chalcraft's conjecture is true. First, there is the issue of choosing which metric to use. In \mathbb{Z}^n or in any hypercube $[\ell]^n$ there are at least two natural choices of a metric: the Euclidean metric $d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ and the graph metric $\sum_{i=1}^n |x_i - y_i|$. Chalcraft's conjecture (for \mathbb{Z}^n) is true for both metrics. Theorem 5 (for $[2]^n$) is independent of the choice of the metric, since if $X, Y \subset \{0, 1\}^n$ are isometric copies with respect to one of the metrics then they are also isometric copies with respect to the other. However, the situation is different in $[\ell]^n$ for $\ell \geq 3$: the obvious version of Chalcraft's conjecture is false for $[\ell]^n$ with the Euclidean metric. For example, take $\ell = 5$ and let $T \subset [5]^2$ be a plus-shaped set of size 5, as shown in Figure 1. Then, no matter what n we choose, it is impossible to partition $[5]^n$ into isometric copies of T because the corners of $[5]^n$ cannot be covered. Similar counterexamples exist for all $\ell \geq 3$.



Figure 1: The plus-shaped set T .

Second, the situation does not become trivial even if we choose the graph metric. It turns out that, with this metric, the obvious version of Chalcraft's conjecture is true for $[\ell]^n$ where $\ell \geq 2$ is even. This fact can be verified in a similar way to Theorem 5; essentially, the only difference is that we have to partition $[\ell]^n$ into copies of $[2]^n$ before we can apply 6 (it is also important to note that $[\ell]^n$ can be isometrically embedded into $[2]^m$ for sufficiently large m). However, the corresponding conjecture would be false for $[\ell]^n$ where $\ell \geq 3$ is odd. Indeed, we will demonstrate that even the corresponding version of the weaker Theorem 4 is false.

We define P_ℓ^n to be the graph with vertex set $[\ell]^n$ where two vertices $(x_1, \dots, x_n), (y_1, \dots, y_n)$ are adjacent if $\sum_{i=1}^n |x_i - y_i| = 1$. We say that a vertex is odd if the sum of its entries is odd; otherwise, that vertex is even.

Proposition 9. *Let $\ell \geq 3$ be an odd integer. Then there exists a graph H satisfying*

- *H is isomorphic to an induced subgraph of P_ℓ^m for some m*
- *$|H|$ is a power of ℓ*
- *for any n , it is impossible to partition the vertices of P_ℓ^n into induced copies of H .*

Proof. Fix an odd integer $\ell \geq 3$ and write A_n and B_n for the number of even and odd vertices in P_ℓ^n , respectively. For any n the graph P_ℓ^n contains a Hamiltonian path, which visits vertices of alternating parity, so we have $|A_n - B_n| \leq 1$. However, $A_n + B_n = |P_\ell^n| = \ell^n$ is odd, so in fact $|A_n - B_n| = 1$. In particular, $A_n \not\equiv 0 \pmod{\ell}$.

Now, choose m sufficiently large so that P_ℓ^m contains an induced connected subgraph on ℓ even and $\ell^2 - \ell$ odd vertices. Denote this subgraph by H . We claim that, for any n , it is impossible to partition the vertices of P_ℓ^n into induced copies of H . Indeed, each induced copy of H in P_ℓ^n contains ℓ or $\ell^2 - \ell$ even vertices. Therefore, the total number of even vertices covered by such a partition would be divisible by ℓ . However, as we saw previously, the number of even vertices in P_ℓ^n is not. \square

It would be interesting to know if Theorem 4 is particular to the hypercubes Q_n or if it holds for powers of other graphs as well. More specifically, let G, H be finite graphs. For any n , we define G^n to be the graph with vertex set $V(G)^n$, where (u_1, \dots, u_n) and (v_1, \dots, v_n) are adjacent if and only if there exists an index $i' \in [n]$ such that $u_i = v_i$ for all $i \neq i'$ and $u_{i'}, v_{i'}$ are adjacent vertices of G . We remark that, with this definition, Q_n is the n th power of the path P_2 consisting of a single edge. What are the natural conditions on H that would make it reasonable to believe that, for some n , G^n can be partitioned into H -sets? Obviously, $|H|$ has to divide $|G|^n$, so we should assume that every prime factor of $|G|$ also divides $|H|$. We should also require H to be isomorphic to an induced subgraph of G^k for some k ; in fact, we may assume that H is isomorphic to an induced subgraph of G itself. However, this is not enough. First, it may still not be possible to cover G^n with copies of H . Moreover, Proposition 9 tells us that even the extra assumption that G can be covered by copies of H would not be enough. After examining why $G = Q_n$ works and $G = P_3^n$ does not, we see that 6 breaks down because P_3^n is not vertex-transitive. We conjecture that Theorem 4 holds whenever we replace Q_n by another vertex-transitive graph.

Conjecture 10. *Let G be a finite vertex-transitive graph and let H be an induced subgraph of G . If every prime factor of $|H|$ divides $|G|$, then there exists a positive integer n such that G^n can be partitioned into induced copies of H .*

What happens if instead of partitioning the vertices of Q_n we attempt to partition the edges? If we want to partition the edge set of Q_n into copies of a fixed graph H , then the obvious necessary divisibility condition is $e(H)2^{n-1}n$, which is satisfied whenever n is a

multiple of $e(H)$. Therefore, as long as H is isomorphic to a subgraph of Q_k for some k , we may expect that such a partition exists for some n . Along with I. Leader and T.S. Tan we make the following conjecture.

Conjecture 11. *Let H be a non-empty subgraph of Q_k for some k . Then there exists a positive integer n such that the edges of Q_n can be covered by edge-disjoint copies of H (the copies of H are not required to be induced).*

It seems to be difficult to prove Conjecture 11 even in very special cases, when we choose H to be a fairly simple graph. For example, we do not know if the conjecture is true when H is Q_k with one edge removed.

On the other hand, the case when H is a path is well understood. Indeed, the edges of Q_n can be partitioned into antipodal paths of the form $(x_1, x_2, \dots, x_n) \rightarrow (1 - x_1, x_2, \dots, x_n) \rightarrow (1 - x_1, 1 - x_2, \dots, x_n) \rightarrow \dots \rightarrow (1 - x_1, 1 - x_2, \dots, 1 - x_n)$ with $x_1 + \dots + x_n$ even. Therefore, $E(Q_n)$ can be partitioned into copies of P_{k+1} whenever n is a multiple of k . Moreover, for odd n , Erde [2] and Anick and Ramras [1] independently determined exactly when $E(Q_n)$ can be partitioned into copies of P_{k+1} : this can be done if and only if $k \leq n$ and $k|2^{n-1}n$. For even n not everything is known yet. Erde conjectured that in this case the obviously necessary conditions $k \leq 2^n$ and $k|2^{n-1}n$ are sufficient.

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References

- [1] D. Anick and M. Ramras, *Edge decompositions of hypercubes by paths*, Australas. J. Combin. **61** (2015), no. 3, 210–226.
- [2] J. Erde, *Decomposing the cube into paths*, Discrete Math. **336** (2014), no. C, 41–45.
- [3] V. Gruslys, I. Leader, and T. S. Tan, *Tiling with arbitrary tiles*, Proc. Lond. Math. Soc. **112** (2016), no. 6, 1019–1039.
- [4] V. Gruslys, I. Leader, and I. Tomon, *Partitioning the boolean lattice into copies of a poset*. Submitted.
- [5] R. W. Hamming, *Error detecting and error correcting codes*, Bell Syst. Tech. J. **29** (1950), no. 2, 147–160.
- [6] Z. Lonc, *Proof of a conjecture on partitions of a boolean lattice*, Order **8** (1991), no. 1, 17–27.

- [7] The Math Forum, *Two tiling problems*, <http://mathforum.org/kb/message.jspa?messageID=6223965> (retrieved on 7 November 2016).
- [8] MathOverflow, *Does every polyomino tile \mathbb{R}^n for some n ?*, <http://mathoverflow.net/questions/49915/does-every-polyomino-tile-rn-for-some-n> (retrieved on 7 November 2016).
- [9] D. Offner, *Packing the hypercube*, Discuss. Math. Graph Theory **34** (2014), no. 1, 85–93.
- [10] M. Ramras, *Symmetric vertex partitions of hypercubes by isometric trees*, J. Combin. Theory Ser. B **54** (1992), no. 2, 239–248.
- [11] D. Rogers, *Unpublished*.
- [12] R. M. Wilson, *Decompositions of complete graphs into subgraphs isomorphic to a given graph*, Congr. number. xv, 1976, pp. 647–659.